Singularities of Theta divisors.

(in abelian varieties)

Abelian varieties: (in particular, "principally polarized")

$$\mathbb{C}^g/\Lambda =: A$$

Λ ← full rank lattice $\cong \mathbb{Z}^{2g}$

Riemann-Bilinear relations:

A is proj. variety if ← "abelian variety."

$$\Lambda \sim \mathbb{Z}^g + \tau\,\mathbb{Z}^g$$

$\sim$: conjugate via $GL(g,\mathbb{C})$

τ: symmetric, complex w/ $\operatorname{im}(\tau)$ pos. def.

Polarization:

$$c \in H^2(A;\mathbb{Z})_{t.f.} \cap H^{1,1}(A;\mathbb{C})$$

is called a polarization for A

if $c = c_1(L)$ for L ample on A.

$\downarrow$

Principality:

c is a principal polarization

iff $h^0(L) = 1$. (well-def because

if $c_1(L) = c_1(L')$

on A, then

$L' = t_a^* L$

some $a \in A$

and $s' \in H^0(L')$

$\parallel$

$t_a^* s$ for $s \in H^0(L)$)

$t_a : A \to A$

$x \mapsto a + x$

c_a's for sell (L)).

c principal $\iff$ $A = \mathbb{C}^g/\Lambda$ has Λ unimodular

$(\Lambda \otimes_{\mathbb{Z}} \Lambda \to \mathbb{Z}$ perfect$)$

Terminology:

Usually choose $\Theta \in |L|$ unique element for $c = c_1(L)$ princ. pol. and write

"(A, Θ) is a p.p.a.v."

princ. pol. ab. var

common parlance: refer to the bundle L as the pol.

common practice: choose $L \in c$ symmetric i.e. so that $(-1)^* L \cong L$.

(2^{2g} ways to choose).

First/main e.g. of ppav:

$$C \rightsquigarrow J(C) = \{\deg 0 \text{ l.b. on } C\} \cong \{\deg g-1 \text{ l.b.}\} \supseteq \{L \mid h^0(L)>0\} = \Theta,$$

$\{L \mid h^0(L)>0\}$ " W_{g-1}, Θ up to transl.

sm. proj. curve $(/\mathbb{C})$ — Jacobian = p.p.a.v.

$$\begin{array}{ccc} C & \rightsquigarrow & J(C) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{M}_g & \xrightarrow{\;J\;} & \mathcal{A}_g \end{array}$$

$\mathcal{M}_g$: curves of genus $g = g(C)$ — dim $3g-3$

$\mathcal{A}_g$: p.p.a.v.s of dim g — dim $\frac{g(g+1)}{2}$

differ for $g \geq 4$.

Begs the question:

Schottky problem: which ppavs are Jacobians of curves?

Fundamental work of Andreotti–Mayer answers the question (to some extent)

A-M: use singularities of Θ to characterize the Jacobians among the p.p.a.v.s.

(essentially "$\dim(\mathrm{Sing}(\Theta)) \geq g-4$

then (A,Θ) came from a curve").

Thm: (Kollár)

(A,Θ) p.p.a.v. is log-canonical.

(in sense of singularities of pairs).

Cor: $\Sigma_k(\Theta) := \{x \in A \mid \mathrm{mult}_x \Theta \geq k\}$

has codim $\geq k$.

Cor: / In particular: $\mathrm{mult}_x \Theta \leq g$.

(any $x \in A$).

<u>Singularities:</u> Kawamata log terminal (klt)

and log-canonical (l.c.)

Log pair $\left[(X, \Delta) \right.$ log pair $\left\{ \begin{array}{l} X \text{ normal} \\ \text{ } \cup\!\text{l} \\ \Delta \text{ eff } \mathbb{Q}\text{-divisor} \\ \text{s.t. } K_X + \Delta \text{ is } \mathbb{Q}\text{-Cartier} \end{array} \right.$

Log res.

$Y \xrightarrow[\varphi]{\text{log res}} X$

1. Y smooth.
2. $\mathrm{exc}(\varphi)$ is <u>divisorial</u>
3. $\tilde{\Delta} \cup \mathrm{exc}(\varphi)$ <u>s.n.c</u> "simple normal crossing"

i.e. locally like int. of coord hyperplanes in $\mathbb{A}^n$

log disc.

Define $\Delta_Y \subseteq Y$ by

1) $K_Y + \Delta_Y = \varphi^*(K_X + \Delta)$

2) $\varphi_* \Delta_Y = \Delta$

Now $E \subseteq Y$ any exc. prime div.

$a_E = a_E(X, \Delta) := 1 - \mathrm{coeff}_E(\Delta_Y)$

(independent of log res.)

Philosophy:

Bigger $a_E \Rightarrow$ "nicer" sing.

("smoother")

?!

Hard conj:

(all $E \subseteq Y$ mapping)

("Nice") $a_E \geq \dim X$ (to pt ex")

$\Rightarrow \Delta = 0$, X smooth.

<u>Defn</u>

(X, Δ) has <u>klt</u> sings iff

nicer → $a_E > 0$ all prime $E \subseteq Y$

" has <u>l.c.</u> sings iff

$a_E \geq 0$ ".

Examples:

1) $\mathbb{P}^1 \cong C \xrightarrow{|\mathcal{O}(n)|} \mathbb{P}^n$ rat'l normal curve of deg n.

$\cap\!|$ $\cap\!|$

$X_n \hookrightarrow \mathbb{P}^{n+1}$

$=$ cone over embedded C

<u>Fact:</u> $Y_n \supseteq E$

$$Y_n \xrightarrow{\varphi} X_n$$

$$Y_n \cong \mathbb{F}_n$$

nth Hirzebruch surface

(ruled surface over $\mathbb{P}^1$

$\cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$)

$$E \cong \mathbb{P}^1$$

$$E^2 = -n.$$

Note:

$$\boxed{K_{Y_n} + \Delta_{Y_n} = \varphi^*(K_{X_n} + \cancel{\Delta})}$$

$$\Rightarrow \Delta_{Y_n} = -K_{Y_n/X_n}.$$

$$= \alpha E \quad \text{some } \alpha \in \mathbb{R}$$

Determining a_E:

Start w/ adjunction on E:

$$(K_{Y_n} + E)\big|_E = K_E.$$

$\downarrow$ take degrees

$$K_{Y_n} \cdot E + \underbrace{E^2}_{(-n)} = 2\cancel{g}^{\,0} - 2$$

$$\Rightarrow K_{Y_n} \cdot E = n-2.$$

$$
\begin{aligned}
(K_{Y_n} + \Delta_{Y_n}) \cdot E &= (K_{Y_n} + \alpha E) \cdot E \\
&= K_{Y_n} \cdot E + \alpha(-n) \\
&= n-2-\alpha n .
\end{aligned}
$$

$$
(K_{Y_n} + \Delta_{Y_n}) \cdot E = \varphi^*(K_X) \cdot E = 0
$$

$$
\Rightarrow \quad \alpha = \frac{n-2}{n} = 1 - \frac{2}{n} .
$$

$$
= \mathrm{coeff}_E(\Delta_{Y_n})
$$

$$
\Rightarrow \quad a_E = 1 - \mathrm{coeff} = \frac{2}{n} .
$$

$$
(\text{large } n \Rightarrow) \quad a_E > 0
$$

$$
\Rightarrow \quad \text{klt}.
$$

replacing C by an elliptic curve (and $\mathcal{O}(n)$ by a deg n l.b. on C)

will result in $a_E = 1 - \frac{n}{n}$

$= 0.$

$\Rightarrow$ l.c. sing.

<u>Thus</u>: first example of klt sing is

Cone (Fano = rat'l norm curve)

while the first e.g. of (strictly) l.c. is

Cone (elliptic normal curve)

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klt / l.c. sings and mult ideals

$$Y \xrightarrow{\mu} X \supseteq D \quad \mathbb{Q}\text{-div} \qquad (X, D).$$

multiplier ideal

$$\mathcal{J}(D) := \mu_*\left(K_{Y/X} - \lfloor \mu^* D \rfloor\right).$$

Prop: (mult ideal detects klt/l.c.)

$$(X,D) \text{ klt} \iff \mathcal{J}(D) = \mathcal{O}_X$$

$$\text{''} \quad \text{l.c.} \iff \mathcal{J}((1-\varepsilon)D) = \mathcal{O}_X$$

for $\varepsilon > 0$ suff. small.

Sketch:

$$\mu^*(K_X + D) = K_Y + D_Y$$
$$\|$$
$$\mu^* K_X + \mu^* D.$$

Weil

$\mathbb{Q}$-div.

$$\Rightarrow \mu^* D = K_{Y/X} + D_Y$$

$$\Rightarrow \lfloor \mu^* D \rfloor = \lfloor \quad \sim\sim \quad \rfloor$$

$$= K_{Y/X} + \lfloor D_Y \rfloor^{\leq 0} \Leftrightarrow \text{klt.}$$

$\Updownarrow$

but $\mathcal{J}(D) = \mu_*(\overbrace{-\lfloor D_Y \rfloor}^{\text{eff}}) = \mathcal{O}_X$.

Remarks: 1) similar computation for the l.c. statement.

2) independent of log resolution.

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rational singularities and adjoint ideals

For adjoint ideals, definitions are analogous to multiplier ideals but w/ following slight diff:

mult ideal:

$X \supseteq D$

needn't be smooth.

$\mathbb{Q}$-Cartier

adjoint ideal:

$$\begin{array}{ccc} X & \supseteq & D \\ \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle\nu} \\ Y & \supseteq & D' \end{array}$$

X smooth. D reduced integral. D' smooth.

$$\mu^* D = D' + F$$

F: μ-exc., eff, int.

Defn:

$$\operatorname{adj}(D) = \mu_*(K_{Y/X} - F).$$

subtract exc, int part instead of round-down.

res. of sings of D

Basic Fact:

i) $0 \to K_X \to (K_X + D) \otimes \mathrm{adj}(D) \to \nu_* \mathcal{O}_{D'}(K_{D'}) \to 0$

ii) $\mathrm{adj}(D) = \mathcal{O}_X \iff D$ normal & rat'l sings.

Theorems on singularities of $\Theta \subseteq A$ p.p.a.v.

(stated earlier) Thm (Kollár):

(A, Θ) has l.c. singularities.

Cor: $\Sigma_k \subseteq \Theta$ has codim $\geq k$.

$\Sigma_k = \{\text{mult} \geq k\}$.

if (A,Θ) splits $\Rightarrow \Sigma_2 \subseteq A$ has codim $= 2$.

(i.e. "$(A_1,\Theta_1)\times(A_2,\Theta_2)$

$$= (A_1 \times A_2, p_1^*\Theta_1 + p_2^*\Theta_2))$$.

Ein-Lazarsfeld: equality in Kollár's theorem must only be a consequence of the ppav (A,Θ) splitting:

Thm (Ein-Laz):

(A,Θ) ppav.

(ii) irreducible

then: (ii) normal

w/ $\leq$ rat'l singularities

Assume Abel's thm:

$$\mathrm{Sym}^k(C) \longrightarrow \mathrm{Pic}^k(C) \cong J(C).$$

$$D \longmapsto \mathcal{O}_C(D).$$

eff deg k divisor

k=g-1:

$$\mathrm{Sym}^{g-1}(C) \longrightarrow \mathrm{Pic}^{g-1}(C).$$

$$W_{g-1} \cong \Theta$$

<u>Proof of Kollár:</u>

What we want to show, for "l.c.ness" of (A, Θ), is triviality of

$$\mathcal{J}((1-\varepsilon)\Theta).$$

FSOC suppose $\mathcal{J}((1-\varepsilon)\Theta) \neq \mathcal{O}_X$

i.e. $\subsetneq \mathcal{O}_X$.

$\Rightarrow$ let $Z = \mathrm{Zeroes}(\mathcal{J}((1-\varepsilon)\Theta))$

subscheme of A

$\Rightarrow$ Have SES

$$0 \to \mathcal{J}_Z \to \mathcal{O}_A \to \mathcal{O}_Z \to 0$$

$$\mathcal{J}_Z = \mathcal{J}((1-\varepsilon)\Theta)$$

$$\left.\begin{array}{l} K_A = \mathcal{O}_A. \\ \text{Nadel vanishing} \end{array}\right\} \Rightarrow H^1(\mathcal{J}_Z(\Theta)) = 0$$

$\Rightarrow$ restr: $H^0(\mathcal{O}_A(\Theta)) \to H^0(\mathcal{O}_Z(\Theta))$

surjective.

BUT recall: $\mathcal{J}(\mathfrak{a}) \supseteq \mathfrak{a}$.

$\Rightarrow$ $Z \subseteq \Theta$

$\Rightarrow$ $s \in H^0(\mathcal{O}_A(\Theta))$ unique (up to scaling) section of princ. pol.
$\|$
defining section of Θ

must vanish on Z.

$\Rightarrow$ restr $= 0$ map.

$\Rightarrow$ $H^0(\mathcal{O}_Z(\Theta)) = 0$

So now the Theorem follows from this Lemma:

Lemma:

(A, Θ) ppav

$Z \subseteq A$ non-empty closed subscheme

$$H^0(\mathcal{O}_Z(\Theta)) \neq 0.$$

Proof: (Uses gp law on A)

for general $a \in A$, the translate $\Theta_a := \Theta + a$ will meet Z properly.

$$\Rightarrow \quad Z \cap \Theta_a = \text{Zeroes}(\sigma)$$

$$\begin{array}{c} \# \\ \phi \end{array} \qquad\qquad \sigma \in H^0(\mathcal{O}_Z(\Theta)).$$

$\square$.

<u>Remark:</u>

Kollár thm: nadel vanishing applied to (mult) ideal sequence.

Ein-Laz: "generic" vanishing applied to adjoint-ideal sequence.

Proof:

Take desing $\nu : X \longrightarrow \Theta$.

$\Rightarrow$ have:

$$0 \longrightarrow K_A \otimes P \longrightarrow \underline{\mathcal{O}_A(\Theta) \otimes \mathrm{adj}(\Theta)} \otimes P \longrightarrow \nu_* \mathcal{O}_X(K_X) \otimes P \to 0$$

$$K_A = \mathcal{O}_A$$

$P \in \mathrm{Pic}^0(A)$.

X has mAd "maximal Albanese dim" $\Rightarrow$ $H^i(K_X + \nu^* P) = 0$, $i > 0$ and generic P.

$\Rightarrow \underline{H^0(\nu_* \mathcal{O}_X(K_X) \otimes P)}$

$= \chi(\mathcal{O}_X(K_X) \otimes \nu^* P)$

$$= \underline{\chi(\mathcal{O}_X(K_X))}.$$

for general P.

Θ irred $\Longrightarrow$ X is of general type.

$h^0(K_X) \leq h^1(\mathcal{O}_A) = g$.

Kawamata - Viehweg $\Rightarrow$ $D \subseteq A$.

$\uparrow$ general type

$\Rightarrow$ $h^0(K_D) \leq g$.

$\shortparallel$

$p_g(D)$

$\Rightarrow \chi(K_X) \neq 0$.

$\Rightarrow$ $\underline{H^0(\mathcal{O}_A(\Theta) \otimes P \otimes \mathrm{adj}(\Theta)) \neq 0}$

$\Rightarrow$ $\mathrm{adj}(\Theta) \neq \mathcal{O}_A$ should follow from a short contrad.

p. 237 in Pos 2.

same trick = translate Θ by gen $a \in A$.